MATH1905 Statistics
Lecture 7

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Lectures: Mon.11am (Chem LT1), Tue.8am (Chem LT1)
Student consultations: Tuesday 2-3pm
General information, tutes, solutions etc...
http://www.maths.usyd.edu.au/u/UG/JM/MATH1905/
or

First Year Office (FYO), Carslaw 520.

Probability: basic concepts

Random experimentTHROW A DICE ONCE (6-faces)

Sample space $\boldsymbol{\Omega}$: list of all possible outcomes (simple events)

$= \{1, 2, 3, 4, 5, 6\}$...if this list is FINITE we denote $|\boldsymbol{\Omega}|$ denotes

the number of simple events in $\boldsymbol{\Omega}$.

Algebra of events:list of all possible subsets of $\boldsymbol{\Omega}$

Let $\boldsymbol{A}, \boldsymbol{B}$ be 2 subsets of $\boldsymbol{\Omega}$. Union:$\boldsymbol{A} \cup \boldsymbol{B} = \boldsymbol{A}$ OR $\boldsymbol{B}$

Intersection: $\boldsymbol{A} \cap \boldsymbol{B} = \boldsymbol{A}$ AND $\boldsymbol{B}$

Mutually exclusive events: $\boldsymbol{A} \cap \boldsymbol{B} = \phi =$ empty set

Example: A=outcome is even B=outcome is odd

$\boldsymbol{A} \cap \boldsymbol{B} = \phi$

Probability: basic concepts

Relative frequencies: THROW A DICE 100 times...record how many 6's=$\boldsymbol{freq(6)}$...$\boldsymbol{P(6) \approx freq(6)/100}$

Statistical regularity: THROW A DICE$\boldsymbol{n}$-times...

...record how many 6's= $\boldsymbol{freq(6)/n \ \longrightarrow_{n\to\infty} 1/6 = P(6)}$

Counting ...when a random experiment has only a FINITE number of possible outcomes,mutually exclusive and equally likely . We can assigne a probability to an event $\boldsymbol{A}$ by counting the number of simple events that corresponds to $\boldsymbol{A}$.If the count is $\boldsymbol{a}$ then we set

$$\boldsymbol{P(A) = \frac{a}{|\Omega|}}$$

Counting

$n!$ = # permutations of n items= # ordered sample of size of n

= # ways to sample n items WITHOUT REPLACEMENT:

$n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$

Example:# ways to draw r ballsWITHOUT REPLACEMENT from an urn which contains n balls:

$n \times (n-1) \times (n-2) \times \ldots \times (n-r+1) = \frac{n!}{(n-r)!}$

$\binom{n}{r}$ (combinations) $= \frac{n!}{(n-r)!r!}$ = # ways of choosing r items among n(without repl.)when ORDER does not counti.e. when choosing 3 numbers amoung 1,2,..,5,6:$\{1,2,3\} = \{2,1,3\}$

Sampling without replacement Suppose a population of size N comprises k different types:

$$\begin{array}{ccccccccc} N & = & N_1 & + & N_2 & + & \ldots & + & N_k \\ & & \text{(type 1)} & & \text{(type 2)} & & & & \text{(type } k\text{)} \end{array}$$

We find the probability that a random sample of size n results in a sample A of the following composition

$$\begin{array}{ccccccccc} n & = & n_1 & + & n_2 & + & \ldots & + & n_k \\ & & \text{(type 1)} & & \text{(type 2)} & & & & \text{(type } k\text{)} \end{array}$$

$$P(A) = \frac{\binom{N_1}{n_1} \times \binom{N_2}{n_2} \times \ldots \times \binom{N_k}{n_k}}{\binom{N}{n}}$$

Example Twelve animals are classified by tail length. There are three with short tails,four with medium and five with long tails.

$N = 12 = 3 + 4 + 5$

(i) If a sample of two is selected at random,what is the probability that one has a short tail and the other a long tail?

$n = 2 = 1 + 0 + 1$

$$P(A) = \frac{\binom{3}{1} \times \binom{4}{0} \times \binom{5}{1}}{\binom{12}{2}} = 0.23(2dp)$$

If six animals are selected at random, what is the probability that tail lengths are equally represented?

$n = 6 = 2 + 2 + 2$

Probability axioms When we do not have a FINITE set of equally likely outcomes...the counting approach is not applicable.However, the realtive frequency of an event tends to stabilise as the sample size increase.We define the probability of an event as the long run relative frequency.We define three axioms:

A1 For any event A, $\mathrm{P}(A) \geq 0$.

...probability can be thought of as a measure of weights (likelihood)

A2 $\mathrm{P}(\Omega) = 1$.

...we scale things so that the largest event Ωhas weight 1

A3 If $A_1, A_2, \ldots, A_n$ are *mutually exclusive* events , then

$$\mathrm{P}\left(A_1 \cup A_2 \cup \ldots \cup A_n\right) = \mathrm{P}(A_1) + \mathrm{P}(A_2) + \cdots + \mathrm{P}(A_n).$$

...when two events A, B (sets) do not overlap the weight of A or B is the sum of the (respective) weights

Consequences **R1** For any event A, $0 \leq \mathrm{P}(A) \leq 1$.
$A \cup A^c = \Omega, \quad P(\Omega) = 1 = P(A) + P(A^c) \geq P(A)$
R2a For arbitrary events A and B,

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B).$$

R2b For **mutually exclusive** events A and B,

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$$

R3 For any event A,$\mathrm{P}(A^c) = 1 - \mathrm{P}(A)$.

Examples

A fair 12–sided die is rolled and the number X on the upper face is noted.Let A be the event that X is a perfect square, B be the event that X is odd and C be the event that X is more than 1.What are the probabilities of the events

$(i)\ A = \{1, 4, 9, \} \quad P(A) = \frac{3}{12} = 0.25$

$(ii)\ B = \{1, 3, 5, ..., 11\}, \quad P(B) = \frac{6}{12} = 0.5$

$(iii)\ C = \{2, 3, 4, 5, ..., 11, 12\}, \quad P(C) = \frac{11}{12} = 1 - \frac{1}{12}$

$(iv)\ A \cap B = \{1, 9\}, \quad P(A \cap B) = \frac{2}{12} = \frac{1}{6}$

A box contains 13 items of which exactly 8 are defective. Four of the items are taken at random from the box. What is the probability that more than two of them are defective? ...

Review problems P&Q pp. 78–79: 1, 2, 3, 5, 6, 8, 9 (a), 9 (b), 11.

R commands

Create an (ordered) list of (integer) numbers: `x=1:12`

Random permutation (sampling without replac.):

```
x.perm=sample(x)
```

Random permutation (sampling without replac,) of size n:

```
x.perm=sample(x,n)
```

Sampling with replac. (bootstrap sampling) of size n:

```
x.boot=sample(x,n,TRUE)
```

Proof of R2a

$A \cup B = (A \cap B^c) \cup (A \cap B) \cup (A^c \cap B)$ $m.e.$ using **A3**

$P(A \cup B) = P(A \cap B^c) + P(A \cap B) + P(A^c \cap B)$

using $A = (A \cap B^c) \cup (A \cap B)$

$P(A) = P(A \cap B^c) + P(A \cap B)$

similarly $P(B) = P(B \cap A) + P(B \cap A^c)$

from which we see that

$P(A \cap B^c) = P(A) - P(A \cap B)$

$P(B \cap A^c) = P(B) - P(B \cap A)$ hence

$P(A \cup B) = P(A) - P(A \cap B) + P(A \cap B) + P(B) - P(B \cap A)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Lecture 8: Conditional probability

Relative frequencies

Hair colour: $A =$ 'fair'; eye colour: $B =$ 'blue'.

	A	A^c	total
B	100	300	400
B^c	200	400	600
total	300	700	1000

The relative frequency of blue-eyed people in the group is $\text{rf}(B) = \frac{400}{1000}$ and the relative frequency of fair-haired, blue-eyed people in the group is $\text{rf}(A \cap B) = \frac{100}{1000}$.
The relative frequency of fair-haired amongst the 400 blue-eyed members of the group is $\text{rf}(A|B) = \frac{100}{400}$.
we can write

$$\text{rf}(A|B) = \text{rf}(A \cap B)/\text{rf}(B)$$

IndependenceTo generalise, write n_{AB} istead of 100,n_B instead of 400,n instead of 1000, and so on.If we consider the attributes of hair and eye colour to be *independent* then we would expect that the relative frequency of fair-haired people amongst all blue eyed people rf$(A|B) = \frac{n_{AB}}{n_B}$should in a large sample be about the same as the relative frequency of fair-haired people amongst the non-blue-eyed,and both should be about the same as the overall relative frequency of fair-haired people, rf$(A) = \frac{n_A}{n}$. Conversely,if rf$(A|B) \approx$ rf(A) in a large sample, it is natural to say that the two attributes are independent.

$\frac{n_{AB}}{n} \approx \frac{n_A}{n} \cdot \frac{n_B}{n}$ This motivates the following definition.

Definition Events A and B are said to be *independent*if

$$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B).$$

Events may be independent under $\mathbf{P}$ but dependent (i.e. not independent) under a different probability, $\mathbf{P}'$, say.(*On the other hand, the mutually exclusivity of events has nothing to do with the choice of probability.*)

ExampleOne red and 19 white balls in an urn.Draw 2 balls at random.$A =$ '1st ball red', $B =$ '2nd ball red'.

Sampling with replacement:

$$\mathrm{P}(A) = \frac{1}{20} \quad \mathrm{P}(B) = \frac{1}{20} \quad \mathrm{P}(B|A) = \frac{1}{20}$$

$$\mathrm{P}(A \cap B) = \mathrm{P}(B|A) \times \mathrm{P}(A) = \frac{1}{20} \times \frac{1}{20} = (\frac{1}{20})^2 = \mathrm{P}(A) \times \mathrm{P}(B)$$

so A and B are independent

Sampling without replacement:

$$\mathbf{P}'(A) = \frac{1}{20} \quad \mathbf{P}'(\text{one ball is red}) = \frac{\binom{19}{1}\times\binom{1}{1}}{\binom{20}{2}} = \frac{2}{20}$$

$$\mathbf{P}'(A) + \mathbf{P}'(B) = \frac{2}{20} \longrightarrow \mathbf{P}'(B) = \frac{1}{20}$$

so $\mathbf{P}'(A) \times \mathbf{P}'(B) = (\frac{1}{20})^2$ but $\mathbf{P}'(A \cap B) = \phi$...

A and B are NOT independent

Independence needs to be checked...

ExampleConsider families with exactly 2 children; suppose events bb, bg, gb, gg are equally likely. Let H denote 'children of both sexes'and A denote 'at most one girl'. Then $\mathrm{P}(A) = 1 - \mathrm{P}(gg) = 3/4$ and $\mathrm{P}(A \cap H) = \mathrm{P}(H) = 2/4$ so A and H are DEPENDENT

Now consider families with 3 children and suppose events bbb, bbg, $\cdots$, ggg are equally likely (1/8 each). Then $\mathrm{P}(A) = \mathrm{P}(\text{no girls}) + \mathrm{P}(\text{one girl}) = \frac{4}{8}$,
$\mathrm{P}(H) = 1 - \mathrm{P}(bbb) - \mathrm{P}(ggg) = \frac{6}{8}$ and $\mathrm{P}(A \cap H) = \frac{3}{8}$ so A and H are INDEPENDENT

Total probability rule. If events B_1, $B_2, \ldots,$ B_n are mutually exclusive and exhaustive (i.e. $\cup_{i=1}^{n} B_i = \Omega$) then for any event A,

$$\mathrm{P}(A) = \sum_{i=1}^{n} \mathrm{P}(A|B_i)\,\mathrm{P}(B_i).$$

Proof for $n = 2$, $B_1 = B$and $B_2 = B^c$,

$$B_1 \cup B_2 = \Omega \quad B_1 \cap B_2 = \phi$$

$$A = (A \cap B^c) \cup (A \cap B)$$

$$P(A) = P(A \cap B^c) + P(A \cap B)$$

$$P(A) = P(A|B^c) \times P(B^c) + P(A|B) \times P(A)$$

ExampleIn a bolt factory, machines A, B and C manufacture 25, 35 and 40 % respectively of the total. The proportions of defectives are 5, 4 and 2 % respectively. What proportion of the produce is defective?

D=1 randomly selected item is defective

$\Omega = A \cup B \cup C$ (1 item is produced by A,B or C)

$$P(D) = P(D|A)\times P(A)+P(D|B)\times P(B)+P(D|C)\times P(C)$$

$$P(D) = 0.05 \times \frac{1}{4} + 0.04 \times \frac{35}{100} + 0.02 \times \frac{40}{100} = 0.0345$$

Bayes rule

$$\mathrm{P}(B_j|A) = \frac{\mathrm{P}(B_j)\mathrm{P}(A|B_j)}{\sum_{i=1}^{n} \mathrm{P}(B_i)\mathrm{P}(A|B_i)}$$

Prooffor $n = 2$, $B_1 = B$and $B_2 = B^c$,

$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A|B) \times P(B)}{P(A)}$
Applying the total probabillity rule

$$P(B|A) = \frac{P(A|B) \times P(B)}{P(A|B^c) \times P(B^c) + P(A|B) \times P(B)}$$

Examples1 In the last example, if a bolt is drawn at random from the produce and found to be defective, what are the probabilities it came from A, B and C?

$$P(A|D) = \frac{P(D|A)\times P(D)}{P(D|A)\times P(A)+P(D|B)\times P(B)+P(D|C)\times P(C)}$$

2 Suppose 5 out of 100 men and 25 out of 10,000 women are colourblind.If a colourblind person is chosen at random from equal numbers of men and women, what is the probability he is male?

$$P(M|B) = \frac{P(B|M)\times P(M)}{P(B|M)\times P(M)+P(B|F)\times P(F)} = \mathbf{0.95}$$

More examples Die A has four red and two white faces, whereas die B has two red and four white faces. A fair coin is flipped once; if it falls heads, the game continues by throwing die A only; if it falls tails, die B is used.

(a) Show the probability of a red is $1/2$ at any throw.
(b) If the first two throws result in red, what is the probability of red at the third throw?
(c) If the first n throws yield red, what is the probability die A is being used?

Review problems P&Q pp. 78–81 7, 9, 14, 23, 24, 25, 26.